PAUL ERDŐS
(1913–1996)

Journal of Approximation Theory **94**, 1–41 (1998)
Article No. AT983254

IN MEMORIAM

Paul Erdős (1913–1996)

1. PROLOGUE

On October 18, 1996, hundreds of people, including many mathematicians, gathered at Kerepesi Cemetery in Budapest to pay their last respects to Paul Erdős. If there was one theme suggested by the farewell orations, it was that the world of mathematics had lost a legend, one of its great representatives. On October 21, 1996, in accordance with his last wishes, Paul Erdős' ashes were buried in his parents' grave at the Jewish cemetery on Kozma street in Budapest.

Paul Erdős was one of this century's greatest and most prolific mathematicians. He is said to have written about 1500 papers, with almost 500 co-authors. He made fundamental contributions in numerous areas of mathematics.

There is a Hungarian saying to the effect that one can forget everything but one's first love. When considering Erdős and his mathematics, we cannot speak of "first love," but of "first loves," and approximation theory was among them. Paul Erdős wrote more than 100 papers that are connected, in one way or another, with the approximation of functions. In these two short reviews, we try to present some of Paul's fundamental contributions to approximation theory.

A list of Paul's papers in approximation theory is given at the end of this article. These are referenced in this article in the form [ab.n], indicating the nth item in the year 19ab. This list is a sublist of the official list of publications by Erdős, in [GN], with a list of additions and corrections available at the website `www.acs.oakland.edu/~grossman/erdoshp.html`. Other references in this article (such as the reference [GN] just used) are listed just prior to that list of Erdős' approximation theory papers.

0021-9045/98 $25.00

Numerous articles and obituaries on Erdős have appeared (see, e.g., the web page www.math.ohio-state.edu/∼nevai/ERDOS/), and more will undoubtedly appear. The interested reader might wish to look at the article by L. Babai which appeared in [Ba].

2. PAUL ERDŐS AND POLYNOMIALS

I will discuss some of Erdős' results related to polynomials that attracted me most. This list reflects my personal taste and is far from complete, even within the subdomains I focus on most, namely polynomial inequalities, Müntz polynomials, and the geometry of polynomials.

The two inequalities below (and their various extensions) play a key role in proving inverse theorems of approximation. Let $\mathcal{P}_n$ denote the set of all algebraic polynomials of degree at most n with real coefficients.

MARKOV'S INEQUALITY. *The inequality*

$$\|p'\|_{L^\infty[-1,1]} \leqslant n^2 \|p\|_{L^\infty[-1,1]}$$

holds for every $p \in \mathcal{P}_n$.

BERNSTEIN INEQUALITY. *The inequality*

$$|p'(y)| \leqslant \frac{n}{\sqrt{1-y^2}} \|p\|_{L^\infty[-1,1]}$$

holds for every $p \in \mathcal{P}_n$ *and* $y \in (-1, 1)$.

For Erdős, Markov- and Bernstein-type inequalities had their own intrinsic interest and he explored what happens when the polynomials are restricted in certain ways. It had been observed by Bernstein that Markov's inequality for monotone polynomials is not essentially better than for arbitrary polynomials. Bernstein proved that *if n is odd, then*

$$\sup_p \frac{\|p'\|_{L^\infty[-1,1]}}{\|p\|_{L^\infty[-1,1]}} = \left(\frac{n+1}{2}\right)^2,$$

where the supremum is taken over all $0 \neq p \in \mathcal{P}_n$ *that are monotone on* $[-1, 1]$. This is surprising, since one would expect that if a polynomial is this far away from the "equioscillating" property of the Chebyshev polynomial, then there should be a more significant improvement in the Markov inequality. In the short paper [40.04], Erdős gave a class of restricted polynomials for which the Markov factor n^2 improves to cn. He proved that there is an absolute constant c such that

$$|p'(y)| \leqslant \min\left\{\frac{c\sqrt{n}}{(1-y^2)^2}, \frac{en}{2}\right\} \|p\|_{L^\infty[-1,1]}, \qquad y \in [-1, 1],$$

for every polynomial of degree at most n that has all its zeros in $\mathbb{R}\backslash(-1, 1)$. This result motivated several people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints. Generalizations of the above Markov- and Bernstein-type inequality of Erdős have been extended in many directions by many people including Lorentz, Scheick, Szabados, Varma, Máté, Rahman, and Govil. Many of these results are contained in the following, due to P. Borwein and T. Erdélyi [BE]: *there is an absolute constant c such that*

$$|p'(y)| \leqslant c \min\left\{\sqrt{\frac{n(k+1)}{1-y^2}}, n(k+1)\right\} \|p\|_{L^\infty[-1,1]}, \qquad y \in [-1, 1],$$

for every polynomial p of degree at most n with real coefficients that has at most k zeros in the open unit disk.

Clarkson and Erdős wrote a seminal paper on the density of Müntz polynomials. Müntz's classical theorem characterizes sequences $\Lambda := (\lambda_i)_{i=0}^\infty$ with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{1}$$

for which the Müntz space $M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, ...\}$ is dense in $C[0, 1]$. Here, $M(\Lambda)$ is the collection of all finite linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, ...$ with real coefficients, and $C(A)$ is the space of all real-valued continuous functions on $A \subset [0, \infty)$ equipped with the uniform norm. If $A := [a, b]$ is a finite closed interval, then the notation $C[a, b] := C([a, b])$ is used.

MÜNTZ'S THEOREM. *Suppose $\Lambda := (\lambda_i)_{i=0}^\infty$ is a sequence satisfying* (1). *Then $M(\Lambda)$ is dense in $C[0, 1]$ if and only if $\sum_{i=1}^\infty 1/\lambda_i = \infty$.*

The point 0 is special in the study of Müntz spaces. Even replacing $[0, 1]$ by an interval $[a, b] \subset [0, \infty)$ in Müntz's theorem is a nontrivial issue. Such an extension is, in large measure, due to Clarkson and Erdős [43.02] and L. Schwartz [Sc]. In [43.02], Clarkson and Erdős showed that Müntz's Theorem holds on any interval $[a, b]$ with $a > 0$. That is, *for any increasing nonnegative sequence $\Lambda := (\lambda_i)_{i=0}^\infty$ and any $0 < a < b$, $M(\Lambda)$ is dense in $C[a, b]$ if and only if $\sum_{i=1}^\infty 1/\lambda_i = \infty$*. Moreover, they described what kind of functions are in the uniform closure of the span on $[a, b]$ assuming $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$. Further, they showed that under the assumption $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ every function $f \in C[a, b]$ from the uniform closure of $M(\Lambda)$ on $[a, b]$ is of the form

$$f(x) = \sum_{i=0}^{\infty} a_i x^{\lambda_i}, \qquad x \in [a, b). \tag{2}$$

In particular, f can be extended analytically throughout the open disk centered at 0 with radius b.

Erdős considered this result his best contribution to complex analysis. Later, by different methods, L. Schwartz extended some of the Clarkson–Erdős results to the case when the exponents λ_i are arbitrary distinct nonnegative numbers. For example, in that case, under the assumption $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ every function $f \in C[a, b]$ from the uniform closure of $M(\Lambda)$ on $[a, b]$ can still be extended analytically throughout the region

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < b\},$$

although such an analytic extension does not necessarily have a representation given by (2). The Clarkson–Erdős results were further extended by Peter Borwein and the author, from the interval $[0, 1]$ to subsets of $[0, \infty)$ with positive Lebesgue measure. That is, *if $\Lambda := (\lambda_i)_{i=0}^{\infty}$ is an increasing sequence of nonnegative real numbers with $\lambda_0 = 0$ and $A \subset [0, \infty)$ is a compact set with positive Lebesgue measure, then $M(\Lambda)$ is dense in $C(A)$ if and only if $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$.* This result had been expected by Erdős and others for a long time.

I find the following result of Erdős and Turán [50.08] especially attractive.

THEOREM. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ has m positive real zeros, then*

$$m^2 \leqslant 2n \log \left(\frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}} \right).$$

This result was originally due to Schur. Erdős and Turán rediscovered it with a short proof.

In [39.02], Erdős proved that the arc length from 0 to 2π of a real trigonometric polynomial f of degree at most n satisfying $|f(\vartheta)| \leqslant 1$ is maximal for $\cos n\vartheta$. An interesting question he posed quite often is the following: *Let $0 < a < b < 2\pi$. Is it still true that the variation and arc-length in $[a, b]$ is maximal for $\cos(n\vartheta + \alpha)$ for a suitable α?* The following related conjecture of Erdős was open for quite a long time: *Is it true that the arc length from -1 to 1 of a real algebraic polynomial of degree at most n is* *maximal for the Chebyshev polynomial* T_n? This was proved independently by Kristiansen [Kr2] and by Bojanov [Boj].

A well-known theorem of Chebyshev states that *if* p *is a real algebraic polynomial of degree at most* n *and* $z_0 \in \mathbb{R}\setminus[-1, 1]$, *then*

$$|p(z_0)| \leqslant |T_n(z_0)| \, \|p\|_{L^\infty[-1, 1]},$$

where T_n *is the Chebyshev polynomial of degree* n. The standard proof of this is based on zero counting which can no longer be applied if z_0 is not real. By letting $z_0 \in \mathbb{C}$ tend to a point in $(-1, 1)$, it is fairly obvious that this result cannot be extended to all $z_0 \in \mathbb{C}$. However, a surprising result of Erdős [47.08] shows that Chebyshev's inequality can be extended to all $z_0 \in \mathbb{C}$ outside the open unit disk.

Erdős and Turán were probably the first to discover the power and applicability of an almost forgotten result of Remez. The so-called Remez inequality is not only attractive and interesting in its own right, but it also plays a fundamental role in proving various other things about polynomials. For a fixed $s \in (0, 2)$, let

$$\mathscr{P}_n(s) := \{ p \in \mathscr{P}_n : m(\{ x \in [-1, 1] : |p(x)| \leqslant 1 \}) \geqslant 2 - s \},$$

where $m(\cdot)$ denotes linear Lebesgue measure. The Remez inequality concerns the problem of bounding the uniform norm of a polynomial $p \in \mathscr{P}_n$ on $[-1, 1]$ given that its modulus is bounded by 1 on a subset of $[-1, 1]$ of Lebesgue measure at least $2 - s$. That is, *how large can* $\|p\|_{L^\infty[-1, 1]}$ (*the uniform norm of* p *on* $[-1, 1]$) *be if* $p \in \mathscr{P}_n(s)$? The answer is given in terms of the Chebyshev polynomials. The extremal polynomials for the above problem are the Chebyshev polynomials $\pm T_n(x) := \pm \cos(n \arccos h(x))$, where h is a linear function which maps $[-1, 1-s]$ or $[-1+s, 1]$ onto $[-1, 1]$.

One of the applications of the Remez inequality by Erdős and Turán [40.05] deals with orthogonal polynomials. Let w be an integrable weight function on $[-1, 1]$ that is positive almost everywhere. Denote the sequence of the associated orthonormal polynomials by $(p_n)_{n=0}^{\infty}$. Then a theorem of Erdős and Turán [40.05] states that

$$\lim_{n \to \infty} [p_n(z)]^{1/n} = z + \sqrt{z^2 - 1}$$

holds uniformly on every closed subset of $\mathbb{C}\setminus[-1, 1]$.

Erdős and Turán [38.05] established a number of results on the spacing of zeros of orthogonal polynomials. One of these is the following.

Let w be an integrable weight function on $[-1, 1]$ *with* $\int_{-1}^{1} (w(x))^{-1}\, dx =: M < \infty$, *and let*

$$(1>)\ x_{1,n} > x_{2,n} > \cdots > x_{n,n}\ (>-1)$$

be the zeros of the associated orthonormal polynomials p_n *in decreasing order. Let*

$$x_{v,n} = \cos \vartheta_{v,n}, \qquad 0 < \vartheta_{v,n} < \pi, \qquad v = 1, 2, ..., n.$$

Let $\vartheta_{0,n} := 0$ *and* $\vartheta_{n+1,n} := \pi$. *Then there is a constant K depending only on M such that*

$$\vartheta_{v+1,n} - \vartheta_{v,n} < \frac{K \log n}{n}, \qquad v = 0, 1, ..., n.$$

This result has been extended by various people in many directions.

Erdős and Freud [74.13] worked together on orthogonal polynomials with regularly distributed zeros. Let α be a nonnegative measure on $(-\infty, \infty)$ for which all the moments

$$\mu_m := \int_{-\infty}^{\infty} x^m\, d\alpha(x), \qquad m = 0, 1, ...$$

exist and are finite. Denote the sequence of the associated orthonormal polynomials by $(p_n)_{n=0}^{\infty}$. Let $x_{1,n} > x_{2,n} > \cdots > x_{n,n}$ be the zeros of of p_n in decreasing order. Let $N(\alpha, t)$ denote the number of positive integers k for which

$$x_{k,n} - x_{n,n} \geqslant t(x_{1,n} - x_{n,n}).$$

The distribution function β of the zeros is defined, when it exists, as

$$\beta(t) = \lim_{n \to \infty} n^{-1} N_n(\alpha, t), \quad 0 \leqslant t \leqslant 1.$$

Let

$$\beta_0(t) = \frac{1}{2} - \frac{1}{\pi} \arcsin(2t - 1).$$

A nonnegative measure α for which the array $x_{k,n}$ has the distribution function $\beta_0(t)$ is called an arc-sine measure. If $d\alpha(x) = w(x)\, dx$ is absolutely continuous and α is an arc-sine measure, then w is called an arc-sine weight. One of the theorems of Erdős and Freud [74.13] states that *the condition*

$$\limsup_{n \to \infty} (\gamma_{n-1})^{1/(n-1)} (x_{1,n} - x_{n,n}) \leqslant 4$$

implies that α *is arc-sine and*

$$\lim_{n\to\infty} (\gamma_{n-1})^{1/(n-1)} (x_{1,n} - x_{n,n}) = 4. \tag{3}$$

They also show that the weights $w_a(x) := \exp(-|x|^a)$, $a > 0$, are not arc-sine. It is further proved by a counterexample that even the stronger sufficient condition (3) in the above-quoted result is not necessary in general to characterize arc-sine measures. As the next result of their paper shows, the case is different if w has compact support. Namely, they show that *a weight* w, *the support of which is contained in* $[-1, 1]$, *is arc-sine on* $[-1, 1]$ *if and only if*

$$\limsup_{n\to\infty} (\gamma_n)^{1/n} \leqslant 2.$$

A set $A \subset [-1, 1]$ is called a determining set if all weights w, the restricted support $\{x: w(x) > 0\}$ of which contains A, are arc-sine on $[-1, 1]$. A set $A \subset [-1, 1]$ is said to have minimal capacity c if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that for every $B \subset [-1, 1]$ having Lebesgue measure less than $\delta(\varepsilon)$ we have $\mathrm{cap}(A \setminus B) > c - \varepsilon$. Another remarkable result of this paper by Erdős and Freud is that *a measurable set* $A \subset [-1, 1]$ *is a determining set if and only if it has minimal capacity* $1/2$.

Erdős' paper [58.05] with Herzog and Piranian on the geometry of polynomials is seminal. In this paper, they proved a number of interesting results and raised many challenging questions. Although quite a few of these have been solved by Pommerenke and others, many of them are still open. Erdős liked this paper very much. In his talks about polynomials, he often revisited these topics and mentioned the unsolved problems again and again. A taste of this paper is given by the following results and still unsolved problems from it. As before, associated with a monic polynomial

$$f(z) = \prod_{j=1}^{n} (z - z_j), \qquad z_j \in \mathbb{C}, \tag{4}$$

let

$$E = E(f) = E_n(f) := \{z \in \mathbb{C}: |f(z)| \leqslant 1\}.$$

One of the results of Erdős, Herzog, and Piranian tells us that *the infimum of* $m(E(f))$ *is* 0, *where the infimum is taken over all polynomials* f *of the form* (4) *with all their zeros in the closed unit disk* (*n varies and m denotes the two-dimensional Lebesgue measure*). Another result is the following: *Let* F *be a closed set of transfinite diameter less than* 1. *Then there exists a positive number* $\rho(F)$ *such that, for every polynomial of the form* (4) *whose zeros lie in* F, *the set* $E(f)$ *contains a disk of radius* $\rho(F)$. There are results on the number of components of E, the sum of the diameters of the components of E, some implications of the connectedness of E, and some necessary assumptions that imply the convexity of E. An interesting conjecture of Erdős states that *the length of the boundary of* $E_n(f)$ *for a polynomial* f *of the form* (4) *is* $2n+O(1)$. This problem seems almost impossible to settle. The best result in this direction is $O(n)$ by P. Borwein [Bor] that improves an earlier upper bound $74n^2$ given by Pommerenke.

One of the papers where Erdős revisits this topic is [73.01], written jointly with Netanyahu. The result of this paper states that *if the zeros* $z_j \in \mathbb{C}$ *are in a bounded, closed, and connected set whose transfinite diameter is* $1-c$ $(0<c<1)$, *then* $E(f)$ *contains a disk of positive radius* ρ *depending only on* c.

Erőd attributes the following interesting result to Erdős and Turán and presents its proof in his paper. *If*

$$f(z) = \pm \prod_{j=1}^{n} (x - x_j), \qquad -1 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant 1, \tag{5}$$

and f *is convex between* x_{k-1} *and* x_k *for an index* k, *then*

$$x_k - x_{k-1} \leqslant \frac{16}{\sqrt{n}}.$$

It is not clear to me whether or not Erdős and Turán published this result.

An elementary paper of Erdős and Grünwald (Gallai) [39.07] deals with some geometric properties of polynomials with only real zeros. One of their results states that *if* f *is a polynomial of the form* (5), *then*

$$\int_{x_k}^{x_{k+1}} |f(x)|\, dx \leqslant \tfrac{2}{3}\,(x_{k+1} - x_k) \max_{x \in [x_k, x_{k+1}]} |f(x)|.$$

Some extensions of the above are proved in [40.02]. In this paper, Erdős raised a number of questions. For example, he conjectured that if t is a real trigonometric polynomial with only real zeros and with maximum 1 then

$$\int_0^{2\pi} |t(\vartheta)|\, d\vartheta \leqslant 4.$$

Concerning polynomials $p \in \mathscr{P}_n$ with all their zeros in $(-1, 1)$ and with $\max_{x \in [-1, 1]} |p(x)| = 1$, Erdős conjectured that *if* $x_k < x_{k+1}$ *are two consecutive zeros of* p, *then*

$$\int_{x_k}^{x_{k+1}} |p(x)|\, dx \leqslant d_n(x_{k+1} - x_k),$$

where

$$d_n := \frac{1}{y_{k+1} - y_k} \int_{y_k}^{y_{k+1}} |T_n(y)| \, dy,$$

T_n is the usual Chebyshev polynomial, and $y_k < y_{k+1}$ are two consecutive zeros of T_n. (Note that d_n is independent of k and that $\lim d_n = 2/\pi$.) These conjectures and more have all been proved in 1974; see Saff and Sheil-Small [SaSh], and also Kristiansen [Kr1].

A paper of Erdős [42.05] deals with the uniform distribution of the zeros of certain polynomials. Let

$$1 = x_0 \geqslant x_1 > x_2 > \cdots > x_n \geqslant x_{n+1} = -1$$

and let $x_i = \cos \vartheta_i$, where $\vartheta_i \in [0, \pi]$. Let $\omega_n(x) = \prod_{i=1}^{n} (x - x_i)$. Let $0 \leqslant A < B \leqslant \pi$. Let $N_n(A, B)$ denote the number of ϑ_i in (A, B). Extending the results of an earlier paper [40.08] of his with Turán, Erdős proved that *if there are absolute constants* $c_1, c_2 > 0$ *such that*

$$\frac{c_1 f(n)}{2^n} \leqslant \max_{x_{k+1} \leqslant x \leqslant x_k} |\omega_n(x)| \leqslant \frac{c_2 f(n)}{2^n}, \quad k = 0, 1, ..., n,$$

then

$$N_n(A, B) = \frac{B - A}{\pi} n + O((\log n)(\log f(n))).$$

The gap condition of Fabry states that if $f(z) = \sum a_k z^{n_k}$ is a power series whose radius of convergence is 1, and $\lim n_k/k = \infty$, then the unit circle is the natural boundary of f. Pólya proved the following converse result. *Let* (n_k) *be an increasing sequence of nonnegative integers for which* $\liminf n_k/k < \infty$. *Then there exists a power series* $\sum a_k z^{n_k}$ *with radius of convergence* 1 *and for which the unit circle is not the natural boundary*. Erdős [45.03] offers a direct and elementary proof of Pólya's result.

Another notable paper of Erdős [47.02], joint with H. Fried, explores some connections between gaps in power series and the zeros of their partial sums. Let $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ be a power series with radius of convergence 1. The power series is said to have Ostrowsky gaps ϱ if there exists a $\varrho < 1$ and a pair of infinite sequences (m_k) and (n_k), with $m_k < n_k$ and $\lim n_k/m_k > 1$ such that $|a_n| < \varrho^n$ for $m_k \leqslant n \leqslant n_k$. Let $A(n, r)$ denote the number of zeros of $S_n(z) := 1 + \sum_{i=1}^{n} a_i z^i$ in the open disk centered at 0 with radius r. A theorem of Erdős and Fried states that *a necessary and* *sufficient condition that a power series have Ostrowsky gaps is that there exists an* $r > 1$ *such that*

$$\liminf \frac{A(n, r)}{n} < 1.$$

Erdős [67.16] gives an extension of some results of Bernstein and Zygmund. Bernstein had asked the question whether one can deduce boundedness of $|P_n(x)|$ on $[-1, 1]$ for polynomials P_n of degree at most n if one knows that $|P_n(x)| \leqslant 1$ for $m > (1+c)n$ values of x with some $c > 0$. His answer was affirmative. He showed that *if* $|P_n(x_i^{(m)})| \leqslant 1$ *for all zeros* $x_i^{(m)}$ *of the mth Chebyshev polynomial* T_m *with* $m > (1+c)n$, *then* $|P_n(x)| \leqslant A(c)$ *for all* $x \in [-1, 1]$, *with* $A(c)$ *depending only on* c. Zygmund had shown that the same conclusion is valid if T_m is replaced by the mth Legendre polynomial L_m. Erdős established a necessary and sufficient condition to characterize the system of nodes

$$-1 \leqslant x_1^{(m)} < x_2^{(m)} < \cdots < x_n^{(m)} \leqslant 1$$

for which

$$|P_n(x_i^{(m)})| \leqslant 1, \qquad i = 1, 2, ..., m; \qquad m > (1+c)n,$$

implies $|P_n(x)| \leqslant A(c)$ for all polynomials P_n of degree at most n and for all $x \in [-1, 1]$, with $A(c)$ depending only on c. His result contains both that of Bernstein and of Zygmund as special cases. Note that such an implication is impossible if $m \leqslant n+1$, by a well-known result of Faber.

Erdős wrote a paper [46.05] on the coefficients of the cyclotomic polynomials. The cyclotomic polynomial F_n is defined as the monic polynomial whose zeros are the primitive nth roots of unity. It is well known that

$$F_n(x) = \prod_{d \mid n} (x^{n/d} - 1)^{\mu(d)}.$$

For $n < 105$, all coefficients of F_n are ± 1 or 0. For $n = 105$, the coefficient 2 occurs for the first time. Denote by A_n the maximum over the absolute values of the coefficients of F_n. Schur proved that $\limsup A_n = \infty$. Emma Lehmer proved that $A_n > cn^{1/3}$ for infinitely many n. In his paper [46.05], Erdős proved that *for every* k, $A_n > n^k$ *for infinitely many* n. This is implied by his even sharper theorem to the effect that

$$A_n > \exp[c(\log n)^{4/3}]$$

for $n = 2 \cdot 3 \cdot 5 \cdots \cdot p_k$ *with* k *sufficiently large*. Recent improvements and generalizations of this can be explored in [Ma1-3].

Erdős has a note [49.08] on the number of terms in the square of a polynomial. Let

$$f_k(x) = a_0 + a_1 x^{n_1} + \cdots + a_{k-1} x^{n_{k-1}}, \qquad 0 \neq a_i \in \mathbb{R},$$

be a polynomial with k terms. Denote by $Q(f_k)$ the number of terms of f_k^2. Let $Q_k := \min Q(f_k)$, where the minimum is taken over all f_k of the above form. Rédei posed the problem whether $Q_k < k$ is possible. Rényi, Kalmár, and Rédei proved that, in fact, $\liminf Q_k/k = 0$, and also that $Q(29) \leqslant 28$. Rényi further proved that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{Q_k}{k} = 0.$$

He also conjectured that $\lim Q_k/k = 0$. In his short note [49.08], Erdős proves this conjecture. In fact, he shows that *there are absolute constants* $c_1 > 0$ *and* $0 < c_2 < 1$ *such that* $Q_k < c_2 k^{1-c_1}$. Rényi conjectured that $\lim Q_k = \infty$. He also asked whether or not Q_k remains the same if the coefficients are complex. These questions remain open (at least in this paper).

Erdős has a number of papers on rational approximation. In [76.20], he proves that *if* f *is a non-vanishing continuous function defined on* $[0, \infty)$ *for which* $\lim_{x \to \infty} f(x) = 0$, *then for every sequence of integers* $0 := n_0 < n_1 < \cdots$ *satisfying* $\sum_{i=1}^{\infty} 1/n_i = \infty$, *there is a sequence of Müntz polynomials* $(p_k) \subset \operatorname{span}\{x^{n_0}, x^{n_1}, ...\}$ *for which*

$$\lim_{k \to \infty} \left\| \frac{1}{f} - \frac{1}{p_n} \right\|_{L^\infty[0, \infty)} = 0. \tag{6}$$

Using a result from the Clarkson–Erdős paper [43.02], he also observes, in [76.20], that if f *is a nonvanishing continuous function defined on* $[0, \infty)$ *for which there exists a sequence* $(p_n) \subset \operatorname{span}\{x^{n_0}, x^{n_1}, ...\}$ *with* $0 := n_0 < n_1 < \cdots$ *and* $\sum_{i=1}^{\infty} 1/n_i < \infty$ *such that* (6) *holds, then* f *is the restriction to* $[0, \infty)$ *of an entire function.*

A typical result of Erdős, Newman, and Reddy [77.04] deals with rational approximations to e^{-x} on $[0, \infty)$. They prove, among many other results, that *if* p *and* q *are real polynomials of degree at most* $n-1$ *with* $n \geqslant 2$, *then*

$$\left\| e^{-x} - \frac{p(x)}{q(x)} \right\|_{L^\infty(\mathbb{N})} \geqslant \frac{(e-1)^n \, e^{-4n} 2^{-7n}}{n(3 + 2\sqrt{2})^{n-1}}.$$

This should be compared with the approximation rate

$$\left\| e^{-x} - \frac{1}{q(x)} \right\|_{L^\infty[0,\infty)} \leqslant 2^{-n}$$

with $q(x) := \sum_{k=0}^{n} x^k/(k!)$. A substantial collection of various results concerning various kinds of rational approximation can be found in another paper of Erdős written jointly with Reddy [76.46].

Erdős [62.01] proved a significant result related to his conjecture about polynomials with ± 1 coefficients. He showed that if

$$f_n(\vartheta) := \sum_{k=1}^{n} (a_k \cos k\vartheta + b_k \sin k\vartheta)$$

is a trigonometric polynomial with real coefficients,

$$\max_{1 \leqslant k \leqslant n} \{\max\{|a_k|, |b_k|\}\} = 1, \qquad \sum_{k=1}^{n} (a_k^2 + b_k^2) = An,$$

then there exists $c = c(A) > 0$ *depending only on* A *for which* $\lim_{A \to 0} c(A) = 0$ *and*

$$\max_{0 \leqslant \vartheta \leqslant 2\pi} |f(\vartheta)| \geqslant \frac{1 + c(A)}{\sqrt{2}} \left(\sum_{k=1}^{n} (a_k^2 + b_k^2) \right)^{1/2}.$$

Closely related to this is a problem for which Erdős offered \$100 and which has become one of my favorite Erdős problems: *Is there an absolute constant* $\varepsilon > 0$ *such that the maximum norm on the unit circle of any polynomial* $p(x) = \sum_{j=0}^{n} a_j x^j$ *with each* $a_j \in \{-1, 1\}$ *is at least* $(1+\varepsilon)\sqrt{n}$? Erdős conjectured that there is such an $\varepsilon > 0$. Even the weaker version of the above, with $(1+\varepsilon)\sqrt{n}$ replaced by $\sqrt{n+\varepsilon}$ with an absolute constant $\varepsilon > 0$, looks really difficult (the lower bound $\sqrt{n+1}$ is obvious by the Parseval formula). Originally, Erdős and D. J. Newman conjectured that there is an absolute constant $\varepsilon > 0$ such that the maximum norm on the unit circle of any polynomial $p(x) = \sum_{j=0}^{n} a_j x^j$ with each $a_j \in \mathbb{C}$, $|a_j| = 1$ is at least $(1+\varepsilon)\sqrt{n}$. An astonishing result of Kahane [K, L] disproves this by showing the existence of "ultra flat" unimodular polynomials with modulus always between $(1-\varepsilon)\sqrt{n}$ and $(1+\varepsilon)\sqrt{n}$ on the unit circle for an arbitrary prescribed $\varepsilon > 0$.

In [65.19], dedicated to Littlewood on his 80th birthday, Erdős gave an interesting necessary condition ensuring that a sequence of integers $0 \leqslant n_0 < n_1 < \cdots$ is not a Zygmund sequence. More precisely, he showed that *if* $0 \leqslant n_0 < n_1 < \cdots$ *is a sequence that contains two subsequences* $(n_{k_i})_{i=1}^{\infty}$ *and* $(n_{l_i})_{i=1}^{\infty}$ *satisfying*

$$k_i \to \infty, \qquad k_i < l_i < k_{i+1}, \qquad l_i - k_i \to \infty, \qquad (n_{l_i} - n_{k_i})^{1/(l_i - k_i)} \to 1,$$

then there is a power series $\sum_{k=0}^{\infty} a_k z^{n_k}$ *with* $|a_k| \to 0$ *that diverges everywhere on the unit circle.* The proof of this theorem utilizes probabilistic arguments which have been used in several earlier papers.

An interesting paper of Erdős [54.07] with Herzog and Piranian deals with sets of divergence of Taylor series and trigonometric series. A typical result of this paper states that *for every subset* E *of the unit circle with logarithmic capacity* 0, *there is a function* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *so that* f *is continuous on the closed unit disk,* $\sum_{n=1}^{\infty} a_n z^n$ *diverges on* E, *and the sequence of partial sums* s_n *is uniformly bounded on the unit circle.*

In 1911, Lusin constructed a power series $\sum_{n=0}^{\infty} a_n z^n$ with $a_n \to 0$ that diverges at every point on the unit circle. Dvoretzky and Erdős [55.05] gave an interesting extension of this result. They proved that *if* $(b_n) \subset \mathbb{C}$ *with* $|b_n| \geqslant |b_{n+1}|$ *for each* n *and* $\sum_{n=0}^{\infty} |b_n|^2 = \infty$, *then there exists a power series* $\sum_{n=0}^{\infty} a_n z^n$ *with each* a_n *equal to either* b_n *or* 0 *that diverges everywhere on the unit circle.* Here the monotonicity condition cannot be entirely dispensed with, since every power series $\sum_{n=0}^{\infty} a_n z^{t_n}$ with $a_n \to 0$ and $\sum_{n=0}^{\infty} t_n / t_{n+1} < \infty$ converges on a subset of the unit circle which is everywhere dense on the unit circle. The condition $\sum_{n=0}^{\infty} |b_n|^2 = \infty$ cannot be relaxed either by Carleson's theorem (Carleson's theorem was a conjecture when Dvoretzky and Erdős wrote their paper, so they commented on this as the above assumption "probably cannot be relaxed at all, since it is conjectured that every power series with $\sum_{n=0}^{\infty} b_n z^n$ with $\sum_{n=0}^{\infty} |b_n|^2 < \infty$ converges almost everywhere in C").

Several topics from Erdős's problem paper [76.14] have already been discussed before. Here is one more interesting group of problems. Let $(z_k)_{k=1}^{\infty}$ be a sequence of complex numbers of modulus 1. Let

$$A_n := \max_{|z|=1} \prod_{k=1}^{n} |z - z_k|.$$

What can one say about the growth of A_n? Erdős conjectured that $\limsup A_n = \infty$. In my copy of [76.14] that Erdős gave me a few years ago, there are some handwritten notes (in Hungarian) saying the following. "Wagner proved that $\limsup A_n = \infty$. It is still open whether or not $A_n > n^c$ or $\sum_{k=1}^{n} A_k > n^{1+c}$ happens for infinitely many n (with an absolute constant $c > 0$). These are probably difficult to answer." [W]

Erdős was famous for anticipating the "right" results. "This is obviously true; only a proof is needed" he used to say quite often. Most of the times, his conjectures turned out to be true. Some of his conjectures failed for the more or less trivial reason that he was not always completely precise with the formulation of the problem. However, it happened only very rarely that he was essentially wrong with his conjectures. If someone proved something that was in contrast with Erdős' anticipation, he or she could really boast to have proved a really surprising result. Erdős was always honest with his conjectures. If he did not have a sense about which way to go, he formulated the problem "prove or disprove." Erdős turned even his "ill fated" conjectures into challenging open problems. The following quotation is a typical example of how Erdős treated the rare cases when a conjecture of his was disproved. It is from his problem paper [76.14] entitled "Extremal Problems on Polynomials." For this quotation, we need to know the following notation: Associated with a monic polynomial $f(z)=\prod_{j=1}^{n}(z-z_j)$, where z_j are complex numbers, let $E_n(f):=\{z\in\mathbb{C}: |f(z)|\leqslant 1\}$. In his problem paper Erdős writes (in terms of the notation employed here): "In [7] we made the ill fated conjecture that the number of components of $E_n(f)$ with diameter greater than $1+c$ $(c>0)$ is less than δ_c, δ_c bounded. Pommerenke [14] showed that nothing could be farther from the truth; in fact he showed that for every $\varepsilon>0$ and $k\in\mathbb{N}$, there is an $E_n(f)$ which has more than k components of diameter greater than $4-\varepsilon$. Our conjecture can probably be saved as follows: Denote by $\Phi_n(c)$ the largest number of components of diameter greater than $1+c$ $(c>0)$ which $E_n(f)$ can have. Surely, for every $c>0$, $\Phi_n(c)=o(n)$, and hopefully $\Phi_n(c)=o(n^{\varepsilon})$ for every $\varepsilon>0$. I have no guess about a lower bound for $\Phi_n(c)$, also I am not sure whether the growth of $\Phi_n(c)$, $(1<c<4)$ depends on c very much."

The list of Erdős' truly ingenious and diverse results concerning polynomials and related topics could be continued for many more pages. One cannot include even all the highlights in a limited space. The reader may correctly think that there are more important results of Erdős in approximation theory than those mentioned in this article. I was concentrating on those results and problems of Erdős that meant the most to me so far and I am looking forward to discovering the beauty in many of his papers that I have not had the chance to read so far.

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3. PAUL ERDŐS AND INTERPOLATION

0. We begin with some definitions and notation. Let $C = C(I)$ denote the space of continuous functions on the interval $I := [-1, 1]$, and let $\mathcal{P}_k$ denote the set of algebraic polynomials of degree at most k. $\|\cdot\|$ will denote the usual uniform norm on C. Let X be an **interpolation array**, i.e.,

$$X = (x_{i,n} = \cos(\vartheta_{i,n}) : i = 1, ..., n;\ n = 0, 1, 2, ...),$$

with

$$-1 \leqslant x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n} \leqslant 1 \tag{1}$$

and $0 \leqslant \vartheta_{i,n} \leqslant \pi$, and consider the corresponding **Lagrange interpolation polynomial**

$$L_n(f, X, x) := \sum_{k=1}^{n} f(x_{k,n})\, \ell_{k,n}(X, x), \quad n \in \mathbb{N}. \tag{2}$$

Here, for $n \in \mathbb{N}$,

$$\ell_{k,n}(X, x) := \frac{\omega_n(X, x)}{\omega_n'(X, x_{k,n})(x - x_{k,n})}, \qquad 1 \leqslant k \leqslant n, \tag{3}$$

with

$$\omega_n(X, x) := \prod_{k=1}^{n} (x - x_{k,n}), \tag{4}$$

are polynomials of exact degree $n-1$. They are called the **fundamental polynomials** associated with the **nodes** $\{x_{k,n} : k = 1, ..., n\}$.

The main question is, of course, the convergence, i.e., to understand for what choices of the interpolation array X we can expect that $L_n(f, X) \to f\ (n \to \infty)$.

Since, by the Chebyshev alternation theorem, the best uniform approximation $P_{n-1}(f)$ to $f \in C$ from $\mathcal{P}_{n-1}$ interpolates f in at least n points, there exists, for each $f \in C$, an interpolation array Y for which

$$\|L_n(f, Y) - f\| = E_{n-1}(f) := \min_{P \in \mathcal{P}_{n-1}} \|f - P\| \tag{5}$$

goes to 0 as $n \to \infty$. However, for the *whole class* C, the situation is much less favourable.

To formulate the corresponding negative result of G. Faber, we quote some estimates and introduce further definitions.

By the classical Lebesgue estimate,

$$\begin{aligned} |L_n(f, X, x) - f(x)| &\leqslant |L_n(f, X, x) - P_{n-1}(f, x)| + |P_{n-1}(f, x) - f(x)| \\ &\leqslant |L_n(f - P_{n-1}, X, x)| + E_{n-1}(f) \\ &\leqslant \left(\sum_{k=1}^{n} |\ell_{k,n}(X, x)| + 1 \right) E_{n-1}(f), \end{aligned} \tag{6}$$

therefore, with the notations

$$\lambda_n(X, x) := \sum_{k=1}^{n} |\ell_{k,n}(X, x)|, \qquad n \in \mathbb{N}, \tag{7}$$

$$\Lambda_n(X) := \|\lambda_n(X, x)\|, \qquad n \in \mathbb{N}, \tag{8}$$

(**Lebesgue function** and **Lebesgue constant** (of Lagrange interpolation), respectively), we have for $n \in \mathbb{N}$

$$|L_n(f, X, x) - f(x)| \leqslant \{\lambda_n(X, x) + 1\}\ E_{n-1}(f) \tag{9}$$

and

$$\|L_n(f, X) - f\| \leqslant \{\Lambda_n(X) + 1\}\ E_{n-1}(f). \tag{10}$$

G. Faber, in 1914, proved the then rather surprising lower bound

$$\Lambda_n(X) \geqslant \tfrac{1}{12} \log n, \qquad n \geqslant 1, \tag{11}$$

for any interpolation array X. Based on this result he obtained (cf. [Fa])

THEOREM 1. (Faber). *For any fixed interpolation array X there exists a function $f \in C$ for which*

$$\overline{\lim_{n \to \infty}}\ \|L_n(f, X)\| = \infty. \tag{12}$$

1. Paul Erdős came to interpolation theory early in his career and wrote on the order of 30 papers in this subject area, quite a few of which were written recently. The general themes considered were:

A. General results on the Lebesgue function and the Lebesgue constant.

B. Construction of "good" nodes for which $L_n(f, X, x) - f(x)$ remains relatively small.

C. Improving Lagrange interpolation by relaxing the degree or replacing the uniform norm with L_p norms.

D. Investigating the divergence behaviour of $L_n(f, X, x)$.

In what follows, we try to sketch some of these results and give an overview of the questions raised, mainly in his more famous problem papers.

2. In the late 1930s, Erdős and Turán wrote three fundamental papers in interpolation theory, [37.04], [38.05], [40.05]. It is difficult to overestimate the impact of these papers. They laid down methods and ideas, and stated theorems on interpolation, orthogonal polynomials and related questions which have been frequently used in the almost 60 years since then. Here we quote some of the theorems.

To state one of the most influential results, we must introduce some further definitions.

Let w be a **weight on** $I=[-1, 1]$, i.e., $w(x) \geqslant 0$ a.e. (almost everywhere) on I, and $\int_I w < \infty$, and let $\{p_n : n \in \mathbb{N}\}$ be the corresponding sequence of orthonormal polynomials (i.e., $\int_I p_n p_m w = \delta_{n,m}$). Then, as is well known, for $n \in \mathbb{N}$, $p_n(w, x) = \gamma_n(w) \prod_{k=1}^n (x - x_{k,n}(w))$ with

$$-1 < x_{n,n}(w) < x_{n-1,n}(w) < \cdots < x_{1,n}(w) < 1, \qquad n \in \mathbb{N}. \tag{13}$$

So, $X(w) := (x_{k,n}(w) \colon 1 \leqslant k \leqslant n, n \in \mathbb{N})$ is an *interpolation* array. Denoting by $L_n(f, w, x)$ the Lagrange polynomial based on (13), Erdős and Turán proved the following theorem highlighting the importance of the $\{p_n\}$ and $X(w)$ in interpolation.

THEOREM 2 [37.04]. *For every $f \in C$ and for every weight w,*

$$\left(\int_{-,1}^{1} |f(x) - L_n(f, w, x)|^2 \, w(x) \, dx \right)^{1/2} \leqslant \sqrt{6} \, E_{n-1}(f). \tag{14}$$

It was only after 40 years (see the works of R. Askey [A] and P. Nevai [N]) that it was shown that the exponent 2 cannot be replaced by $2+\varepsilon$ for *every* weight.

Continuing on this theme, Erdős and Feldheim proved the following surprising result for the weight $(1-x^2)^{-1/2}$.

THEOREM 3 [36.10]. *For every $f \in C$ and $0 < p < \infty$*

$$\lim_{n \to \infty} \int_{-1}^{1} |f(x) - L_n(f, T, x)|^p \frac{1}{\sqrt{1-x^2}} \, dx = 0. \tag{15}$$

Here $T=(t_{k,n}:=\cos[(2k-1)/2n]\pi\colon 1\leqslant k\leqslant n, n\in\mathbb{N})$ is the Chebyshev array, with $T_n(x)=:\gamma_n\prod_{k=1}^n(x-t_{k,n})$ the Chebyshev polynomials (they tend to show good behaviour in interpolation questions). The corresponding result for trigonometric interpolation based on equidistant nodes was proved independently by J. Marcinkiewicz [M1].

3. For $n\in\mathbb{N}$, the **Hermite–Fejér (HF) interpolation polynomial** $H_n(f,X,x)$ is the unique polynomial in P_{2n-1} which satisfies the conditions

$$\begin{cases} H_n(f,X,x_{k,n})=f(x_{k,n}), & 1\leqslant k\leqslant n,\\ H_n'(f,X,x_{k,n})=0, & 1\leqslant k\leqslant n.\end{cases} \tag{16}$$

As L. Fejér (teacher and advisor of Erdős) proved in 1916, the HF polynomials do converge to f for good choices of the array X. Namely, Fejér proved that

THEOREM 4 [Fe2]. *For every* $f\in C$,

$$\lim_{n\to\infty}\|H_n(f,T,x)-f(x)\|=0. \tag{17}$$

Compare this result with Theorem 1 and note that $L_n\in\mathcal{P}_{n-1}$ while $H_n\in\mathcal{P}_{2n-1}$!

It easily follows that

$$H_n(f,X,x)=\sum_{k=1}^n f(x_{k,n})\,h_{k,n}(X,x), \tag{18}$$

where $h_{k,n}\in\mathcal{P}_{2n-1}$, the **fundamental functions of first kind of HF interpolation,** have the form

$$\begin{aligned} h_{k,n}(X,x) &:= \left\{1-\frac{\omega_n''(X,x_{k,n})}{\omega_n'(X,x_{k,n})}(x-x_{k,n})\right\}\ell_{k,n}^2(X,x)\\ &=: v_{k,n}(X,x)\,\ell_{k,n}^2(X,x). \end{aligned} \tag{19}$$

If there exists a positive fixed number ϱ so that the linear functions $v_{k,n}$, defined in (19), satisfy the inequalities

$$v_{k,n}(X,x)\geqslant\varrho>0, \qquad 1\leqslant k\leqslant n, \quad n\in\mathbb{N}, \quad x\in I, \tag{20}$$

then Fejér called the array X **ϱ-normal**. For example, if $P_n^{(\alpha,\beta)}(x)=c_n^{(\alpha,\beta)}\prod_{k=1}^n(x-x_{k,n}^{(\alpha,\beta)})$ $(\alpha,\beta>-1)$ is the nth Jacobi polynomial, then $X^{(\alpha,\beta)}:=(x_{k,n}^{(\alpha,\beta)})$ is a ϱ-normal interpolation array with $\varrho=\min(-\alpha,-\beta)$

whenever $-1 < \alpha, \beta < 0$. Thus $T = X^{(-1/2, -1/2)}$, the Chebyshev array, is 1/2-normal.

In 1942, G. Grünwald showed the significance of ϱ-normal arrays by proving

THEOREM 5 (Grünwald [G3]). *If X is ϱ-normal, then for all $f \in C$*

$$\lim_{n \to \infty} \| H_n(f, X, x) - f(x) \| = 0. \tag{21}$$

No wonder Erdős, who was a doctoral student of Fejér, turned to HF interpolation and ϱ-normality.

In the paper [38.05], Erdős and Turán proved some (as they called them) Fejérian theorems. Using certain properties of the Lagrange and HF fundamental polynomials, they were able to obtain results on the distribution of the nodes. We mention only the two most frequently quoted results.

THEOREM 6 [38.05]. *If X is an interpolation array and*

$$\| \ell_{k,n}(X, x) \| \leqslant c \tag{22}$$

for every k and n, then

$$\vartheta_{k+1,n} - \vartheta_{k,n} \sim \frac{1}{n}, \qquad k = 1, 2, ..., n-1. \tag{23}$$

Moreover, condition (22) can be replaced by

$$\| h_{k,n}(X, x) \| \leqslant c \tag{24}$$

for every k and n.

4. The paper [40.05], written in 1939, is "dedicated to Professor L. Fejér on the occasion of his sixtieth birthday". It considers, among other matters, interpolation based on ϱ-normal arrays and on the roots of orthogonal polynomials.

A typical theorem is the following.

THEOREM 7 [40.05]. *If X is ϱ-normal, then*

$$\| \omega_n(X, x) \| \leqslant \frac{8}{\sqrt{\varrho}} \frac{\sqrt{n}}{2^n}. \tag{25}$$

The authors give two relatively simple proofs of this result.

With regard to orthogonal polynomials, this paper deals with four basic problems: behaviour on the interior of $[-1, 1]$, behaviour on the exterior of $[-1, 1]$, the distance between consecutive roots, and the distribution of roots.

We should also mention Lemma 4 of that paper which turned out to be fundamental in investigating $\lambda_n(X, x)$.

THEOREM 8 [40.05, Lemma 4]. *Let $X = (x_{k,n})$ in $[-1, 1]$ be an arbitrary interpolation array. Then*

$$\ell_{k,n}(X, x) + \ell_{k+1,n}(X, x) \geqslant 1, \qquad x \in [x_{k+1,n}, x_{k,n}], \qquad 1 \leqslant k \leqslant n-1. \tag{26}$$

5. One of Erdős' famous co-authors in interpolation theory was the previously mentioned Hungarian mathematician G. Grünwald. G. Grünwald was a holocaust victim; he was killed in 1943 at the age of 33. Erdős and Grünwald wrote three joint papers. In the paper [38.01], they proved

THEOREM 9 [38.01]. *Let T be the Chebyshev array. Then*

$$|\ell_{k,n}(T, x)| < \frac{4}{\pi}$$

for all k and n, and

$$\lim_{n \to \infty} |\ell_{1,n}(T, 1)| = \frac{4}{\pi}.$$

This result was subsequently generalized by Erdős in [44.05].

G. Grünwald (1935, 1936) and J. Marcinkiewicz (1937) proved the following result.

THEOREM 10 (Grünwald [G1; G2], J. Marcinkiewicz [M2]). *There exists a function $f \in C$ for which*

$$\overline{\lim_{n \to \infty}} |L_n(f, T, x)| = \infty \tag{27}$$

for every $x \in [-1, 1]$.

In their second joint paper, [38.12], Erdős and Grünwald sharpen this result. They construct a function $f \in C$ satisfying (27), where, at the same time, the even function $f(\cos \vartheta)$ has a uniformly convergent Fourier series on $[0, \pi]$.

Marcinkiewicz [M2] showed that for every x_0 there exists a continuous f for which

$$\varlimsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} L_k(f, T, x_0) = \infty. \tag{28}$$

In other words, the arithmetic means of the Lagrange interpolating polynomials of a continuous function can diverge at a given point. This is in marked contrast to the celebrated theorem of Fejér [Fe1] for Fourier series.

In their third joint paper, [37.09], Erdős and Grünwald claimed to prove a far reaching generalization of (28), namely the existence of an $f \in C$ for which

$$\varlimsup_{n\to\infty} \frac{1}{n} \left| \sum_{k=1}^{n} L_k(f, T, x) \right| = \infty, \tag{29}$$

for all $x \in [-1, 1]$. However, as was discovered later by Erdős himself, there is an oversight in the proof and the method only gives this result with the modulus sign *inside* the summation.

Only in [91.23] were Erdős and G. Halász (who was born four years later than the Erdős–Grünwald paper) able to complete the proof and obtain the following result.

THEOREM 11 [91.23]. *Given a positive sequence* (ε_n) *converging to zero however slowly, one can construct a function* $f \in C$ *such that for almost all* $x \in [-1, 1]$

$$\frac{1}{n} \left| \sum_{k=1}^{n} L_k(f, T, x) \right| \geqslant \varepsilon_n \log\log n \tag{30}$$

for infinitely many n.

The right-hand side is optimal, for in the paper [50.13] Erdős has proved that

THEOREM 12 [50.13].

$$\frac{1}{n} \left| \sum_{k=1}^{n} L_k(f, T, x) \right| = o(\log\log n) \tag{31}$$

for almost all x, *whenever* $f \in C$.

The proof of (30) was an ingenious combination of ideas from number theory, probability and interpolation. It is not by chance that the authors are Erdős and Halász!

6. Estimates (9)–(12) show clearly the importance of the Lebesgue function $\lambda_n(X, x)$ and the Lebesgue constant $\Lambda_n(X)$. During the last 40 years Erdős proved very general relations concerning their behaviour and applied them to obtain the strongest possible divergence theorems for $L_n(f, X)$.

To begin, let us state the counterpart of (11). Namely, using an estimate of L. Fejér's [Fe3] (cf. [T, Section 4.12.6]),

$$\Lambda_n(T) = \frac{2}{\pi} \log n + O(1), \tag{32}$$

one can see that the order $\log n$ in (11) is best possible.

S. Bernstein (1931) (cf. [B]) improved upon (11) by showing that for any fixed X one can choose a point $x_0 \in [-1, 1]$ for which

$$\lambda_n(X, x_0) > \left\{\frac{2}{\pi} + o(1)\right\} \log n. \tag{33}$$

In his paper [58.16], Erdős proved the following statement which turned out to be crucial.

Theorem 13 [58.16, Lemma 3]. *Let $y_1, y_2, ..., y_t$ be any t $(t > t_0)$ distinct numbers in $[-1, 1]$ not necessarily in increasing order. Then, for at least one j $(1 \leqslant j \leqslant t)$,*

$$\sum_{i=1}^{u-1} \frac{1}{|y_i - y_j|} > \frac{t \log t}{8}. \tag{34}$$

(The half-page proof is based on the inequality between the arithmetic and harmonic means.)

Inequalities (34) and (26) were used to obtain the main result of [58.16], namely:

Theorem 14 [58.16]. *For any fixed interpolation array $X \subset [-1, 1]$, real $\varepsilon > 0$, and $A > 0$, there exists $n_0 = n_0(A, \varepsilon)$ so that the set*

$$\{x \in \mathbb{R}: \lambda_n(X, x) \leqslant A \text{ for all } n \geqslant n_0(A, \varepsilon)\} \tag{35}$$

has measure less than ε.

Almost 20 years ago, and 20 years after the above result, Erdős and J. Szabados proved:

THEOREM 15 [78.29]. *For an arbitrary interpolation array X and any fixed interval $[a, b] \subset I$,*

$$\int_a^b \lambda_n(X, x)\, dx \geqslant c(b-a) \log n, \quad n \geqslant n_0(a, b), \tag{36}$$

where $c > 0$ is an absolute constant.

We remark that many of the ideas of this paper were used and developed to obtain pointwise estimates for $\lambda_n(X, x)$ (cf. (38)) and for the **incomplete Lebesgue function** (i.e., when in (7) we exclude the index k whenever $x_{k,n}$ is "too close" to x). Such pointwise estimates are of fundamental importance in the a.e. divergence theorem (cf. part 8 below).

Recently Erdős, P. Vértesi and J. Szabados, [95.01], refined (36) by showing that

$$\int_{a_n}^{b_n} \lambda_n(X, x) dx \geqslant c(b_n - a_n) \log\{n(\alpha_n - \beta_n) + 2\}, \tag{37}$$

where $a_n = \cos \alpha_n$, $b_n = \cos \beta_n$. This result can handle small intervals whose lengths may depend on n.

The pointwise estimate (35) was improved upon more than 20 years later by P. Erdős and P. Vértesi. They proved the following.

THEOREM 16 [81.16]. *Let $\varepsilon > 0$ be given. Then, for any fixed interpolation array $X \subset [-1, 1]$, there exist sets $H_n = H_n(\varepsilon, X)$ of measure $\leqslant \varepsilon$ and a number $\eta = \eta(\varepsilon) > 0$ such that*

$$\lambda_n(X, x) > \eta \log n \tag{38}$$

if $x \in [-1, 1] \backslash H_n$ and $n \geqslant 1$.

Closer investigation shows that (instead of the original $\eta = c\varepsilon^3$) $\eta = c\varepsilon$ can be attained (cf. [V1]). The behaviour of the Chebyshev array T shows that (38) is best possible in order.

Now note the significant gap in the lower bounds (11) and (32), between the constants $1/12$ and $2/\pi$. In 1961, strongly using the ideas and results of the papers [40.08] and [42.05], Erdős and Turán, [61.01], proved that they can replace (11) by

$$\Lambda_n(X) \geqslant \frac{2}{\pi} \log n - c \log \log n. \tag{39}$$

In the same year, using a different and rather delicate argument, Erdős [61.20] proved that log log n can be replaced by c; namely

$$\Lambda_n(X) \geqslant \frac{2}{\pi} \log n - c. \tag{40}$$

This is a very significant and strong result as, by (32), it then follows that

$$\left| \Lambda_n^* - \frac{2}{\pi} \log n \right| \leqslant c, \tag{41}$$

where

$$\Lambda_n^* := \min_{X \subset I} \Lambda_n(X), \qquad n \geqslant 1, \tag{42}$$

is the **optimal Lebesgue constant**. As a consequence of this result, the closer investigation of Λ_n^* attracted the attention of many mathematicians. The problem turned out to be rather difficult.

We had to wait almost another 20 years to get a better estimate for Λ_n^*. First, in 1978, T. Kilgore, C. de Boor, and A. Pinkus proved the so-called Bernstein-Erdős conjectures concerning the optimal interpolation array X (cf. [Ki; BP; BrP]). Their result was then applied in a series of papers to get Λ_n^* within the error $o(1)$ (L. Brutman, R. Güntter, P. Vértesi, and others [Br], [Gü1], [Gü2], [V2], [V3]).

7. One of the most significant contributions of Erdős to interpolation theory is his joint paper [55.09] with Paul Turán dedicated to L. Fejér on his 75th birthday.

In the class Lip α $(0 < \alpha < 1)$, a natural error estimate for Lagrange interpolation is

$$\|L_n(f, X) - f\| \leqslant cn^{-\alpha} \Lambda_n(X)$$

(cf. (10)). Erdős and Turán raised the following natural question: *How sharp is this estimate in terms of the order of the Lebesgue constant as $n \to \infty$?* They themselves considered interpolation arrays X, where

$$\Lambda_n(X) \sim n^{\beta} \qquad (\beta > 0).$$

(In the class Lip α, this is a natural setting.) In the above paper [55.09] they obtained the following results.

THEOREM 17 [55.09]. *Let X be as above. If $\alpha > \beta$, then we have uniform convergence in* Lip α. If $\alpha < \beta/(\beta + 2)$, *then, for some $f \in$* Lip α, *Lagrange interpolation is divergent* (*in fact, $\|L_n(f, X)\|$ is unbounded*).

These two cases comprise what is called the "rough theory," since solely on the basis of the order of $\Lambda_n(X)$ one can decide the convergence-divergence behavior. However,

THEOREM 18 [55.09]. *If* $\beta/(\beta+2)<\alpha<\beta$, *then anything can happen. That is, there is an interpolation array* Y_1 *with* $\Lambda_n(Y_1)\sim n^\beta$ *and a function* $f_1\in Lip\ \alpha$, *such that* $\overline{\lim}_{n\to\infty}\|L_n(f_1, Y_1)\|=\infty$, *and another interpolation array* Y_2 *with* $\Lambda_n(Y_2)\sim n^\beta$, *such that* $\lim_{n\to\infty}\|L_n(f, Y_2)-f\|=0$ *for every* $f\in \mathrm{Lip}\ \alpha$.

That is, to decide the convergence–divergence behavior, we need more information than just that given by the Lebesgue constant. The corresponding situation is called "fine theory" (similar results were stated without proof by S. M. Lozinskii in 1948 [Lo]).

This paper of Erdős and Turán has been very influential. It left open a number of problems and attracted the attention not only of the Hungarian school of interpolation (G. Freud, O. Kis, M. Sallay, P. Vértesi, J. Szabados), but also of others as well (R. J. Nessel, W. Dickmeis, E. van Wickeren). Quite recently, G. Halász [H] gave a new interpretation and generalization to the original results of Erdős and Turán.

8. After the result of Grünwald and Marcinkiewicz (cf. (27)), a natural problem was to obtain an analogous result for an *arbitrary* array X. In [58.16, p. 384], Erdős wrote: "In a subsequent paper I hope to prove the following result:

Let $X\subset[-1, 1]$ *be any point group [interpolation array]. Then there exists a continuous function* $f(x)$ *so that for almost all* x

$$\overline{\lim_{n\to\infty}}\ |L_n(f, X, x)|=\infty.\text{"}$$

After 4 long years of work, Erdős and Vértesi proved the above result, [80.25], [81.03], [81.12]. Erdős writes in [80.25]: "[Here we prove the above] statement in full detail. The detailed proof turns out to be quite complicated and several unexpected difficulties had to be overcome."

In a personal letter, Erdős wrote about the main idea of the proof: [First] "we should prove that for every fixed A and $\eta>0$ there exists an M ($M=M(A, \eta)$) such that if we divide the interval $[-1, 1]$ into M equal parts $I_1,..., I_M$ then

$$\sum_k{}' \ |\ell_{k,n}(X, x)|>A, \qquad x\in I_r,$$

apart from a set of measure $\leqslant\eta$. Here $\sum'$ means that k takes those values for which $x\notin I_r$."

9. L. Fejér's result (17) shows that if the degree of the interpolation polynomial is about two times bigger than the number of interpolation points, then we can get convergence. Erdős raised the following question. Given $\varepsilon > 0$, suppose we interpolate at n nodes, but allow polynomials of degree at most $n(1+\varepsilon)$. Under what conditions will they converge for all continuous functions?

The story of this problem is typically Erdősian. In [43.06], Erdős stated an answer to the above problem, but instead of proving it, he just gave an indication that "the proof is a simple modification of Theorem 3." After some 45 years, as a result of the joint effort of Erdős, Kroó, and Szabados, the original statement concerning the above problem was proved, even in a slightly stronger form. The result is the following.

THEOREM 19 [89.16]. *For every $f \in C$ and $\varepsilon > 0$, there exists a sequence of polynomials $p_n(f)$ of degree at most $n(1+\varepsilon)$ such that*

$$p_n(f, x_{k,n}) = f(x_{k,n}), \qquad 1 \leqslant k \leqslant n,$$

and

$$\|f - p_n(f)\| \leqslant c E_{\lfloor n(1+\varepsilon) \rfloor}(f)$$

for some $c > 0$, holds if and only if

$$\limsup_{n\to\infty} \frac{N_n(I_n)}{n\,|I_n|} \leqslant \frac{1}{\pi} \qquad \textit{whenever} \quad \lim_{n\to\infty} n\,|I_n| = \infty \tag{43}$$

and

$$\lim_{n\to\infty} n \min_{1\leqslant k\leqslant n-1} (\vartheta_{k+1,n} - \vartheta_{n,k}) > 0. \tag{44}$$

Here, $N_n(I_n)$ is the number of the $\vartheta_{k,n}$ in $I_n \subset I$. Condition (43) ensures that the nodes are not too dense, and condition (44) says that adjacent nodes should not be too close.

10. One of the immediate consequences of the Faber theorem, (11), is the following:

For any interpolation array X, there exists a polynomial $P \in \mathscr{P}_{n-1}$ for which

$$|P(x_{k,n})| \leqslant 1, \qquad 1 \leqslant k \leqslant n, \tag{45}$$

but

$$\|P\| \geqslant \tfrac{1}{12} \log n. \tag{46}$$

Erdős [67.16], inspired by some results of S. Bernstein and A. Zygmund, asked the following question (quoted here in the notation used in this survey).

"*Let X be an interpolation array. What are necessary and sufficient conditions on X that if for* $P \in \mathscr{P}_{\lfloor n(1-\varepsilon)\rfloor}$ ($\varepsilon > 0$ *is arbitrarily fixed*) *and*

$$|P(x_{k,n})| \leqslant 1, \quad 1 \leqslant k \leqslant n,$$

then

$$\|P\| \leqslant c(\varepsilon)$$

should hold?"

In his fairly difficult paper [67.16] Erdős finds the answer. Loosely speaking, he proves that the distance between two consecutive roots cannot be too small (i.e., $o(1/n)$) on the circle and the density of the interpolation array cannot exceed the density of the roots of the Chebyshev polynomials. The exact formulation and other relevant questions—which are also closely connected to the previous papers [43.06] and [89.16]—are left to the interested reader (cf. (49) and the related problem).

11. In place of (7) one can investigate the expression $\lambda_n(s)(X, x) := \sum_{k=1}^{n} |\ell_{k,n}(X, x)|^s$ for arbitrary interpolation array X. Erdős claimed that for any interpolation array X

$$\int_{-1}^{1} \lambda_n(2)(X, x)\, dx \geqslant 2 - c\,\frac{\log n}{n}$$

but never came up with a proof. Recently in a joint paper, Erdős with J. Szabados, A. K. Varma and P. Vértesi proved a weaker statement, namely:

THEOREM 20 [94.08]. *For any interpolation,*

$$\int_{-1}^{1} \lambda_n(2)(X, x)\, dx \geqslant 2 - c\,\frac{\log^2 n}{n}, \qquad n \geqslant 1. \tag{47}$$

A similar result holds if we estimate the weighted integral of $\lambda_n(2s)(X, x)$.

12. A paper on Erdős is not complete without mention of his exceptional ability to pose fundamental, interesting, and sometimes quite difficult problems. He created a new form of mathematical writing in his famous problem-papers. We finish this part with an extended sample.

Erdős, "the absolute monarch of problem posers," wrote some dozen problem-papers on his and his fellow mathematicians' questions and results on interpolation. Many of the questions posed have been solved, others seem too difficult to handle.

Erdős writes (cf. [91.27, pp. 253–257]):

"In the 1930s G. Grünwald and J. Marcinkiewicz proved that there is a continuous function $f(x)$ for which the interpolation polynomials taken at the roots of the Chebyshev polynomials $T_n(x)$ diverge everywhere in $-1 \leqslant x \leqslant 1$. This was an important and surprising result and several of us tried to extend it to the Fourier expansion. 30 years later Carleson showed that for the Fourier expansion we have convergence almost everywhere. About 10 years ago Vértesi and I [80.25] proved that for every point group [*interpolation array* X] there is a continuous function $f(x)$ for which $L_n(f, X, x)$ diverges almost everywhere. The proof is quite difficult and is best possible, since it is not difficult to find point groups for which $L_n(f, X, x)$ converges for a set of power c. Now the following interesting problem can be posed: Is there a point group for which for every continuous function there is at least one x_0 for which

$$\overline{\lim_{n \to \infty}}\ \lambda_n(X, x_0) = \infty$$

holds but $L_n(f, X, x_0) \to f(x_0)$? In other words $L_n(f, X)$ cannot be completely bad, i.e., it cannot diverge simultaneously at all points where divergence is possible. In fact perhaps there are $c = 2^{\aleph_0}$ such points. At the moment I do not see how to attack this problem, which probably is quite difficult. The classical results of Grünwald and Marcinkiewicz show that the roots of $T_n(x)$ do not have this property.

The following problem also seems very difficult: Is it true that for every point group there is a continuous function $f(x)$ for which the arithmetic means of the Lagrange interpolation polynomials

$$\frac{1}{n} \sum_{k=1}^{n} L_k(f, X, x) \tag{48}$$

diverge almost everywhere [cf. (30)]...? This if true would be a strengthening of the classical result of Grünwald–Marcinkiewicz. The following question seems interesting but is perhaps difficult: Is there a point group for which the following strange behaviour holds: There is a set A in $[-1, +1]$ for which there is a continuous function $f(x)$ for which $L_n(f, X, x)$ diverges everywhere in A, but this is not possible for the arithmetic means (48)? I proved 40 years ago that in some sense the arithmetic means behave better than $L_n(f, X, x)$. In case of the roots of $T_n(x)$, $L_n(f, T, x)$ can diverge as fast as $o(\log n)$ but the sequence (48) can diverge only as fast as $o(\log \log n)$. This result though no doubt quite interesting throws no light on our problem just formulated ([50.13]).

Let $x_0 = \cos(p/q)\pi$, $p \equiv q \equiv 1 \pmod 2$. I proved that if we interpolate on the roots of $T_n(x)$ then there is a continuous $f(x)$ for which (see [41.02] and [43.03])

$$|L_n(f, T, x_0)| \to \infty.$$

I claimed that a slight modification gives a continuous $f(x)$ for which $L_n(f, T, x) \to \infty$ and in fact I claimed that for any closed set A there is a continuous $f(x)$ for which the set of limit points of $L_n(f, T, x)$ is precisely the set A. I feel it would be worthwhile to work out the proof in detail....

I think the following conjecture (if true) would be of great interest. It would be a significant extension of the old results of Faber and Bernstein and of our theorem with Vértesi and it would show that our theorem with Kroó and Szabados is best possible in a different sense [Cf. [80.25] and [43.06], [89.16]].

Let $\{x_{k,n}\}$ be an arbitrary point group, and let $\varepsilon_n \to 0$ as slowly as we please. Is it true that there always is a continuous $f(x)$ for which $p_n(x)$ is a polynomial of degree $< n(1+\varepsilon_n)$ which satisfies

$$p_n(x_{k,n}) = f(x_{k,n}), \qquad 1 \leqslant k \leqslant n \tag{49}$$

then $p_n(x)$ will diverge almost everywhere? In other words a sequence of polynomials which satisfies (49) will diverge almost everywhere.

As far as I know this conjecture has not even been proved if the $x_{k,n}$ are given by the roots of $T_n(x)$.

Some results which seem to point in the direction of this conjecture were proved by Shekhtman and Szabados [Cf. [Sh], [Sz]]... .

I proved [(40)] that

$$\Lambda_n(X) > \frac{2}{\pi} \log n - c,$$

and Kilgore [with the help of de Boor and Pinkus] settled an old conjecture of S. Bernstein by proving that $\Lambda_n(X)$ is minimal if all the $n+1$ maxima of $\sum_{k=1}^{n} |\ell_k(X, x)|$ are equal. This beautiful result leaves the following question unanswered: Let $\{x_{k,n}\}$ be a point group. Is it true that there is an x_0, $-1 \leqslant x_0 \leqslant 1$ for which

$$\varlimsup_{n \to \infty} \sum_{k=1}^{n} |\ell_k^{(n)}(X, x_0)| > \frac{2}{\pi} \log n - c, \tag{50}$$

and in fact (50) perhaps holds for almost all x_0? I claimed [61.20], [68.16] that I proved that for every $-1 \leqslant a < b \leqslant 1$

$$\max_{a \leqslant x \leqslant b} \sum_{k=1}^{n} |\ell_k(X, x)| > \left|\frac{2}{\pi} + o(1)\right| \log n. \tag{51}$$

The proof of (51) was never published and I think if it would be reconstructed it would deserve publication. Szabados and I proved that [78.29]

$$\int_a^b \lambda_n(X, x)\, dx > c(b-a) \log n,$$

but we could not determine the best value of c. We conjectured that the best value of c is given if the x_i are the roots of $T_n(x)$." (Cf. (37), too.)

Here is another problem from the classical theory of interpolation. Erdős conjectured that the quantity

$$\int_{-1}^{1} \sum_{k=1}^{n} \ell_{k,n}^2(X, x)\, dx \tag{52}$$

attains its minimum if and only if the nodes are the roots of the integral of the Legendre polynomials. He had a good reason for this; it is well known that the minimum of

$$\left\| \sum_{k=1}^{n} \ell_{k,n}^2(X, x) \right\| \tag{53}$$

is attained if and only if the nodes are the ones mentioned above, and in this case (51) is equal to 1. Nevertheless, in 1966 Szabados disproved this conjecture; so the characterization of the minimizing system of nodes is still unsolved.

We end this part with a list of some papers of Erdős which are exceptionally rich in problems: [50.13], [55.09], [58.16], [61.20], [67.16], [68.16], [80.25], [83.24] and [91.27].

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4. A PERSONAL MEMORY OF PAUL ERDŐS

Paul Erdős (Erdős Pál) died on September 20, 1996 in Warsaw. He was 83. It is safe to assume that he was trying to solve problems on his last day as he was almost every minute in his life. His life is chronicled very nicely by Paul Hoffman in the November 1987 *Atlantic Monthly*, and even more so by L. Babai in [Ba]. Mathematics was his life and he was a connoisseur of problem solving. He learned mathematics by solving problems. He is known to be the most prolific mathematician in history. It is impossible to imagine a person who was more devoted to mathematics. He gave up almost everything that one can enjoy in life for the joy of exploring mathematics by solving problems. He saw life in terms of mathematics, and (jokingly?) called a small kid “ε,” husbands “slaves,” wives “bosses,” people with family “captured,” and so on. Erdős never “died.” “Died” was the word he used to mean “stopped doing math.” Probably he would say he “left” or, as Euler in his famous story, he “finished.” I recall a comment of his when he was once angry with a secretary. He left the office and turned to me, saying “she does not have a chance to prove the Riemann Hypothesis.” As a Hungarian, quite frequently he was thinking in terms of Hungarian expressions. I guess that most of his jokes and funny expressions were created in Hungarian first. They simply sound better and more funny in Hungarian, and a word for word English translation would sound rather silly. Sometimes in his lectures in the U.S.A. he turned to one of the Hungarians in the audience and repeated the sentence in Hungarian after saying “it sounds better in Hungarian.”

He visited our Mathematics Department at Texas A&M University in February 1996 and gave a sequence of talks. Our distinguished visitors typically give three talks, one for the general faculty, one for specialists, and one for graduate students. His talks were also advertised accordingly. I thought it was a mistake. He always presented talks that were equally interesting to students and mathematicians with good taste. In fact, he formed the taste of his audience and the readers of his papers. In his talks, he presented mathematics as a useful, living, evolving, and human discipline. If I were asked to name the person who formed my mathematical taste the most, next to my advisors at various levels of my study, I would mention Erdős, despite the fact that we met only a few times. He conveyed the richness and vitality of mathematics by attending to its social and historical dimensions. Every lecture of his contained a number of memorable short stories and anecdotes, and he always gave plenty to take home. He had an amazing memory. Many times his preparation for a talk was simply organizing his thoughts without any notes. Before his second talk in our department, he was talking with me throughout the morning. Someone came by the office and frantically said that Erdős’ talk would start in five minutes. Erdős talked to me for two more minutes, then looked at his watch and said "In three minutes I have a lecture, I have to prepare."

He often joked about his age and loss of memory. In fact, he had an amazing memory even at the age of 83. He frequently remembered the exact time and place (sometimes even the weather) when and where some good idea had occurred to him. Nevertheless I recall a funny story that happened during his visit at Texas A&M. In one of our discussions, he asked who proved a certain result about polynomials. I told him it was Pommerenke. He kept thinking and half a minute later he said: "No, it was not Pommerance, it was Pommerenke." An hour later, he said in his lecture: "One who could explain how the brain works would certainly deserve a Nobel Prize. An hour ago, I asked someone in my office who proved this theorem. He told me it was Pommerance, and from this I almost immediately recalled that it was Pommerenke."

Erdős was supposed to receive an honorary degree at Texas A&M. He finished his last talk of his visit at Texas A&M by saying jokingly: "I will come back in December to receive an honorary degree, assuming both I and the university exist." I thought that it was my bad luck that due to my visit to Copenhagen for the year I would miss this chance to accompany Erdős during his stay. Unfortunately, I did not miss my last chance.

Most mathematicians understood his obsession to some extent, although most likely no one understood it completely. For mathematics, he sacrificed things that most people think necessary to live a happy life. Most people outside mathematics found him very strange. He owned barely anything. He had no home, no family, no job, and no permanent address. He did not need these. What he needed was the maximal amount of time for thinking about mathematical problems. If he proved a major result in the morning, in the afternoon he could think about another longstanding conjecture. In fact, it always seemed to me that he did not even want to waste any time to stop for a few days and just reflect and enjoy the beauty of his achievements. He was never worried about financial security. He knew that he was getting enough for his talks to be able to reach his next destination. Not doing mathematics was a waste of time for him, and he did not want anything that could have possibly prevented him from doing mathematics. His friends and colleagues realized this and took care of the affairs of everyday life for him such as checks, tax returns, transportation, food, and the like. Sometimes he behaved like a lost child and he was naively honest in showing whether or not he was interested in what his partner was saying. He was genuinely interested in the mathematical work of almost everybody, but he requested people to summarize their best achievements briefly. If he found the summary too detailed, he simply changed topics and started to talk about other things. This was another way in which he formed the mathematical taste of his colleagues. However, I never found this attitude of his insulting. I have to admit that mostly he was right about it, and I learned a great deal from him in this way.

His mother was very close to Erdős and traveled with him a lot. Sometimes in our conversations, he suddenly started to talk about her. For example, he mentioned to me on at least two different occasions that once he arrived at Vancouver with his mother on a foggy day, and in the fog and rain she did not understand why her son described the city as one of the most beautiful cities on Earth. The next day was sunny, and she agreed.

Although his primary interest was always mathematics, he was quite informed about other things in life. Once, he was sitting next to me during a dinner in his honor. Hearing that I was talking about soccer with a Polish colleague of mine, he stopped writing a formula on a napkin and said "How come that we Hungarians are so weak in soccer recently? We used to be much better. Can you explain it?" I tried to present my "theory," and to my great surprise he was genuinely interested in it and he shared his opinions with me. He also said "I occasionally watch soccer games on television. I clearly remember that I watched England beating Germany in 1966 at Lake Balaton with Paul Turán" (he appeared not to know that this match was the World Cup Final in 1966).

From September 1985 to June 1987, I was working in the Mathematics Institute of the Hungarian Academy of Sciences in Budapest. I had a number of discussions with Erdős during this period. Typically he would sit in the office of the director and have discussions with more than a dozen people a day. He particularly liked discussions with students and young mathematicians. He gave various pieces of advice, directions, suggestions, references, hints, ideas, or he simply expressed his feelings as to whether or not a conjecture was true. These were often combined with short stories and historical remarks. It was very easy to approach him. In fact, most of the time, he initiated the discussions. He always wanted to know about the problems that people in the Institute were working on. Usually I was working in my office when I received calls from Erdős from the office of the director. He was so busy with seeing people that sometimes by the time I reached his office he had time only to arrange a time for our next discussion. I had a few lunches with him in the restaurants nearby. I can recall an occasion when I had lunch early but, when Erdős asked me to have lunch with him, I pretended that I was still hungry. Somehow a number of people must have known the place where he had lunch each day because our conversations were often interrupted by phone calls reaching him at the restaurant.

In January 1987, I attended one of my first mathematical conferences, in Havana, Cuba. In one of our discussions, Erdős asked for details about this trip and at some point he started to worry that my (modest) financial support from the Academy would not be sufficient to pay the hotel expenses in Havana. “If you run out of money, just ask my good friend Saff to give you some, and tell him that I will pay it back soon,” he said. At that time, I did not even know Ed Saff and as he told me years later, he doubted that he would have given money to a complete stranger even if he referred to Erdős. Fortunately I did not need to take advantage of this offer. However, this little story always reminds me of the genuinely giving and caring spirit of Erdős.

Erdős had far more co-authors than any other mathematician, ever. I believe one does not need to define “Erdős number” in this journal. There are Erdős number trees available on the Internet (see `www.acs.oakland.edu/~grossman/erdoshp.html`). According to one of these, he had 485 co-authors (that is, 485 mathematicians are listed with Erdős number 1). He was in constant contact with his collaborators. He had many calls and wrote several letters almost every day. In January, 1993, I received a card from him. He wrote it in Hungarian. It translates as “Haifa is a nice city. Let $f(z)=\prod_{j=1}^{n}(z-z_j)$ where z_j are complex numbers. Let $E=E(f):=\{z\in\mathbb{C}\colon |f(z)|\leqslant 1\}$. Is it true that $\operatorname{diam}(E)\geqslant 2$? It must be trivial but I cannot see this at the moment.” It turned out that he was absolutely right about this. The answer is yes, and it is not that difficult to prove. This qualifies for being trivial in the vocabulary of Erdős.

The statement *the mathematician is a machine that converts coffee into theorems* is incorrectly attributed to Erdős. I am not really a coffee drinker, but during Erdős’ visit to College Station, I was not able to avoid drinking coffee with him a few times. I found out that he himself attributed the above saying to Rényi. He added that he just agreed with it.

Erdős was not known as a person who gave practical advice to people about how to live. “You absolutely need to learn how to drive in the U.S.A.” was one of the last pieces of advice he gave me on February 1 in College Station. This was the last day I saw him. I talked to him one more time. He called me the next day around midnight from Dallas. “Erdélyi? This is Erdős Pál” he said. “I found the paper containing an example for a function on $[0,1]$ that is not monotone on any uncountable subset of $[0,1]$” he continued and gave me the reference (Sierpinsky, *Fundamenta Math.* **1** (1923), 316–318) while I was turning off the television showing a hockey game. A former classmate of mine from Hungary had e-mailed this question to me, and I had answered him half-jokingly that “Erdős is just visiting us so an answer may be given soon.” I was right about it. “I am happy that we sorted this out. Good night!” These were the last words I heard from him. Of course, as always, he was talking in Hungarian with Hungarians. Whenever he visited a place throughout the world, he always looked for Hungarians. I sensed that he particularly enjoyed having Hungarians around him.

He has contributed to many important fields of mathematics. Once I heard a comment that described Erdős' contribution to mathematics as a diamond broken into a thousand pieces. Would Erdős be considered as an even more significant mathematician if he had focused only on a few major problems? People can argue about it, but I think the term "broken into" is unfair. However, a thing that everybody must agree about Erdős is that his contribution to mathematics can be described as a great many pieces of diamonds.

Several of these pieces are in or related to the field of approximation theory, in particular polynomial inequalities, the geometry of polynomials, and interpolation. Nevertheless, Erdős' greatness lay in the fact that he was never thinking in terms of "fields." He was thinking in terms of "naturally interesting" problems. There are many results of his related to approximation theory that have some flavor of probability, number theory, potential theory, combinatorics, ergodic theory, geometry, and optimization (the list could be continued). In Section 2 of this survey, I have mentioned only some of the highlights of his research in approximation theory. The highlights I selected are quite personal. Different people might choose different results to demonstrate the richness and vitality of Erdős' work in approximation theory. This is a question of taste. Problems of Erdős' about polynomials always looked particularly attractive and challenging to me. His taste had a tremendous impact on me. He loved to give talks not only for experts but also for high-school, undergraduate, and graduate students. During my stay in the Mathematics Institute in Budapest between 1985 and 1987, I discussed many problems with him concerning inequalities for polynomials with restricted zeros. His mathematical taste represented everything that made me want to be a mathematician.

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ACKNOWLEDGMENTS

The authors are grateful for the editors' help in coordinating their efforts. This material is based upon work supported by the National Science Foundation under Grant DMS–9623156 (T.E.) and by the Hungarian National Foundation for Scientific Research under Grants T7570, T22943, and T17425 (P.V.).

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